Andrey Tarasov

Introduction to QCD

Lecture 1

PROGRAM TOPICS WILL INCLUDE:

- **Introduction to QCD** — Andrey Tarasov (Jefferson Lab, USA)
- **Parton Distribution Functions** — Amanda Cooper-Sarkar (U. of Oxford, UK)
- **TMDs and Quantum Entanglement** — Christine Aidala (U. of Michigan, USA)
- **Nucleon Spatial Imaging** — Julie Roche (Ohio U, USA)
- **QCD and Hadron Structure** — Marcus Diehl (DESY, Germany)
- **Effective Field Theories** — Emilie Passemar (Indiana U., USA)
- **Neutron Skins in Nuclei** — Jorge Piekarewicz (Florida State U., USA)

Overview of the course

Lecture 1: QCD at different scales

Introduction to QCD. QCD Lagrangian. Color.
Perturbation theory. Running of the coupling constant.

Lecture 2: QCD at tree level

Deep inelastic scattering. Parton model.

Lecture 3 and 4: QCD at one loop

Radiative correction in the parton model

Lecture 5: QCD and evolution equations

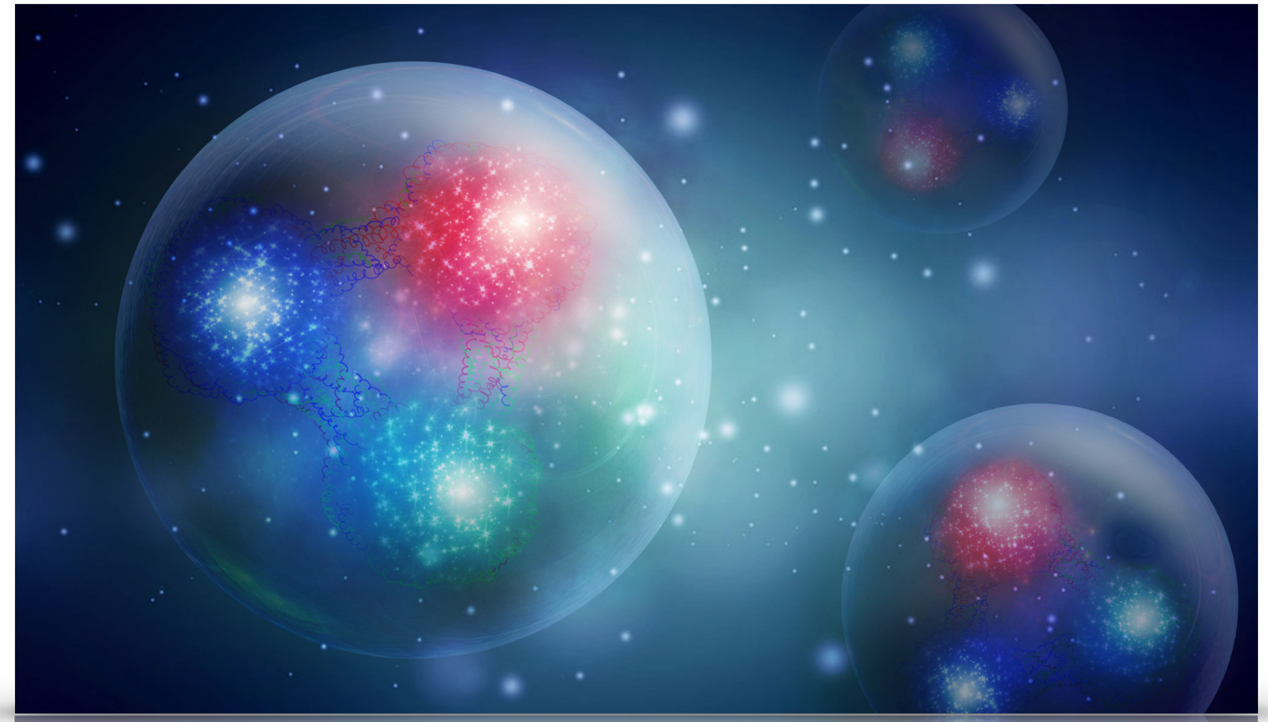
DGLAP evolution equation.

Lecture 6: Introduction to small-x

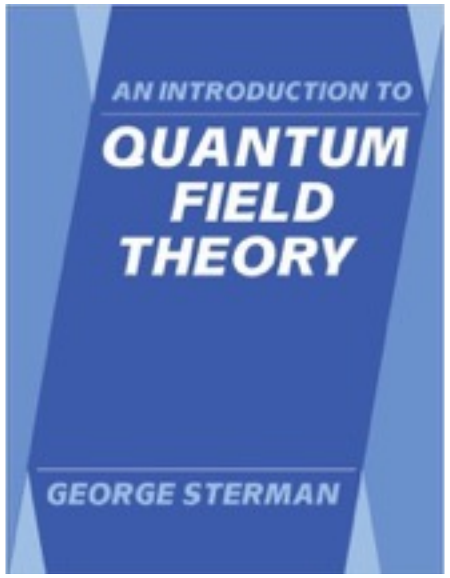
Introduction to high-energy QCD

We will actually do QCD calculations

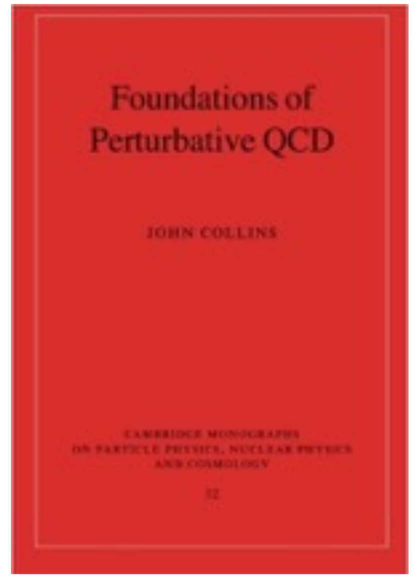
QCD is a window to the world of quarks and gluons



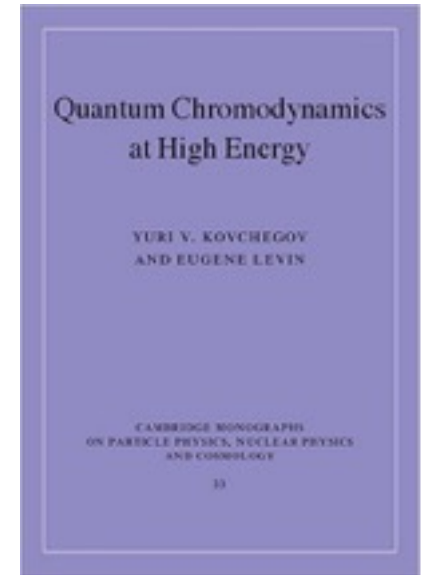
Textbooks



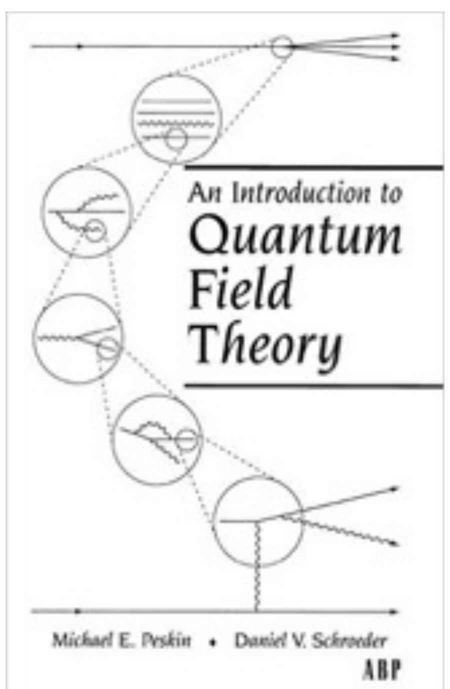
An Introduction to Quantum Field Theory by George Sterman



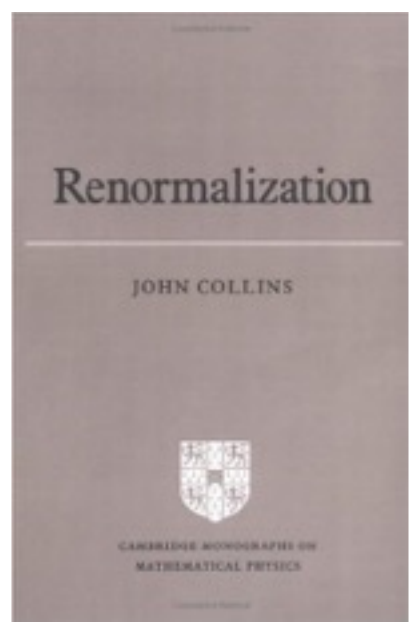
Foundations of Perturbative QCD by John Collins



Quantum Chromodynamics at High Energy by Y.V. Kovchegov and E. Levin



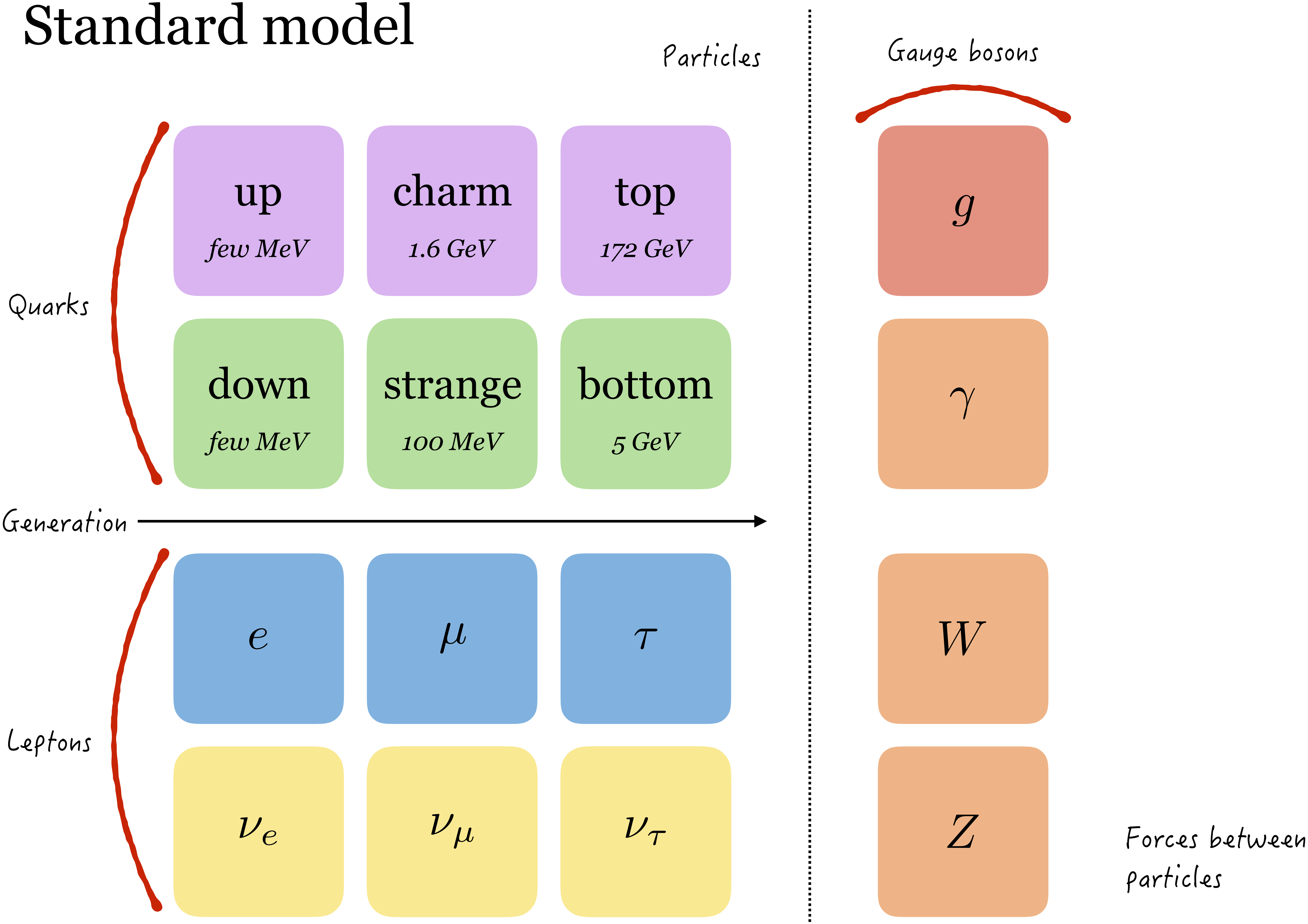
An Introduction to Quantum Field Theory by M.E. Peskin and D.V. Schroeder



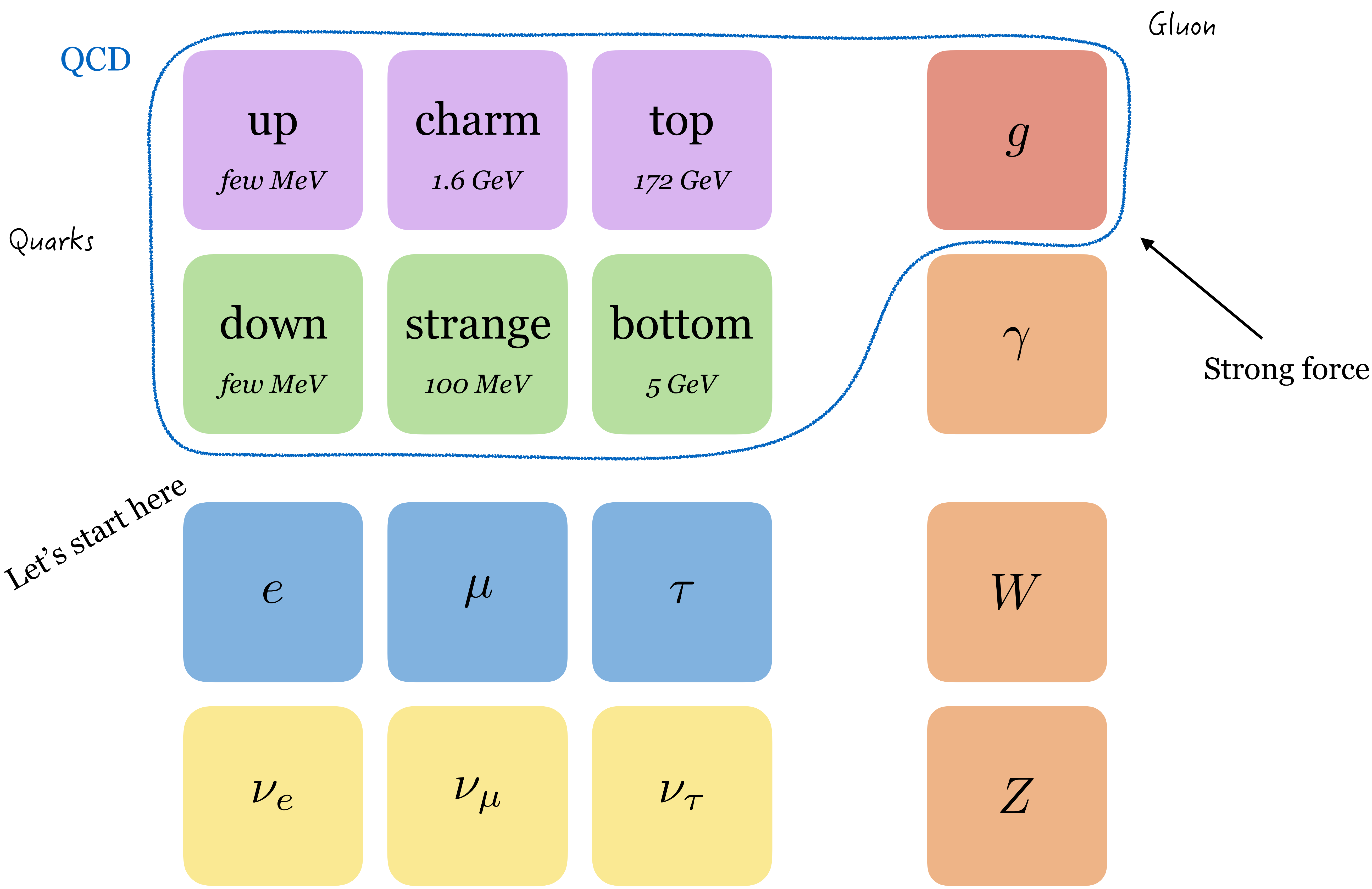
Renormalization by John Collins

To understand QCD you have to make calculation by yourself

Standard model



Quantum chromodynamics (QCD)



QED Lagrangian

$$\mathcal{L}_{QED} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}^2$$

Dirac part. Describes leptons.

Maxwell part. Describes photons.

$$\psi_i = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

$$D_\mu = \partial_\mu + ieA_\mu$$

$$A_\mu = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix}$$

Interact through covariant derivative (gamma matrices)

$$\not{D} = \gamma^\mu_{ij} D_\mu$$

Dirac index

This is a field (operator)

Lorentz index

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Field strength tensor

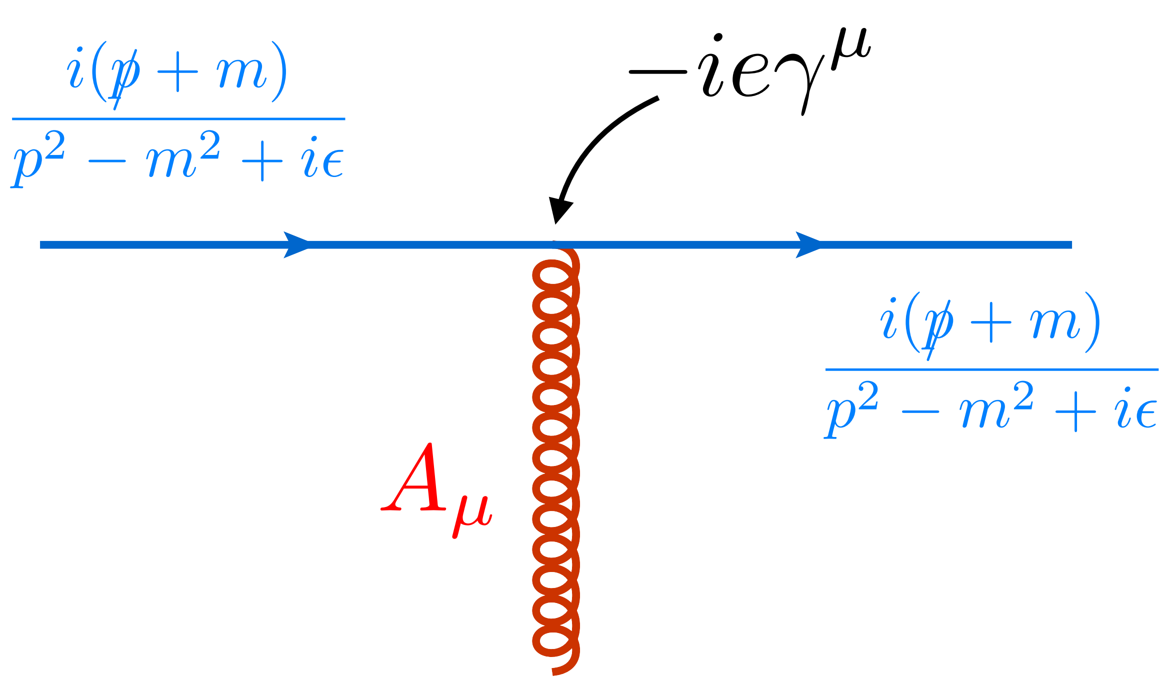
QED Lagrangian: interaction term

$$\mathcal{L}_{QED} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}F_{\mu\nu}^2 - e\bar{\psi}_i\gamma^{\mu}_{ij}\psi_j A_\mu$$

quark (free part)

photon (free part)

interaction



We don't need to know explicit form of gamma matrix.

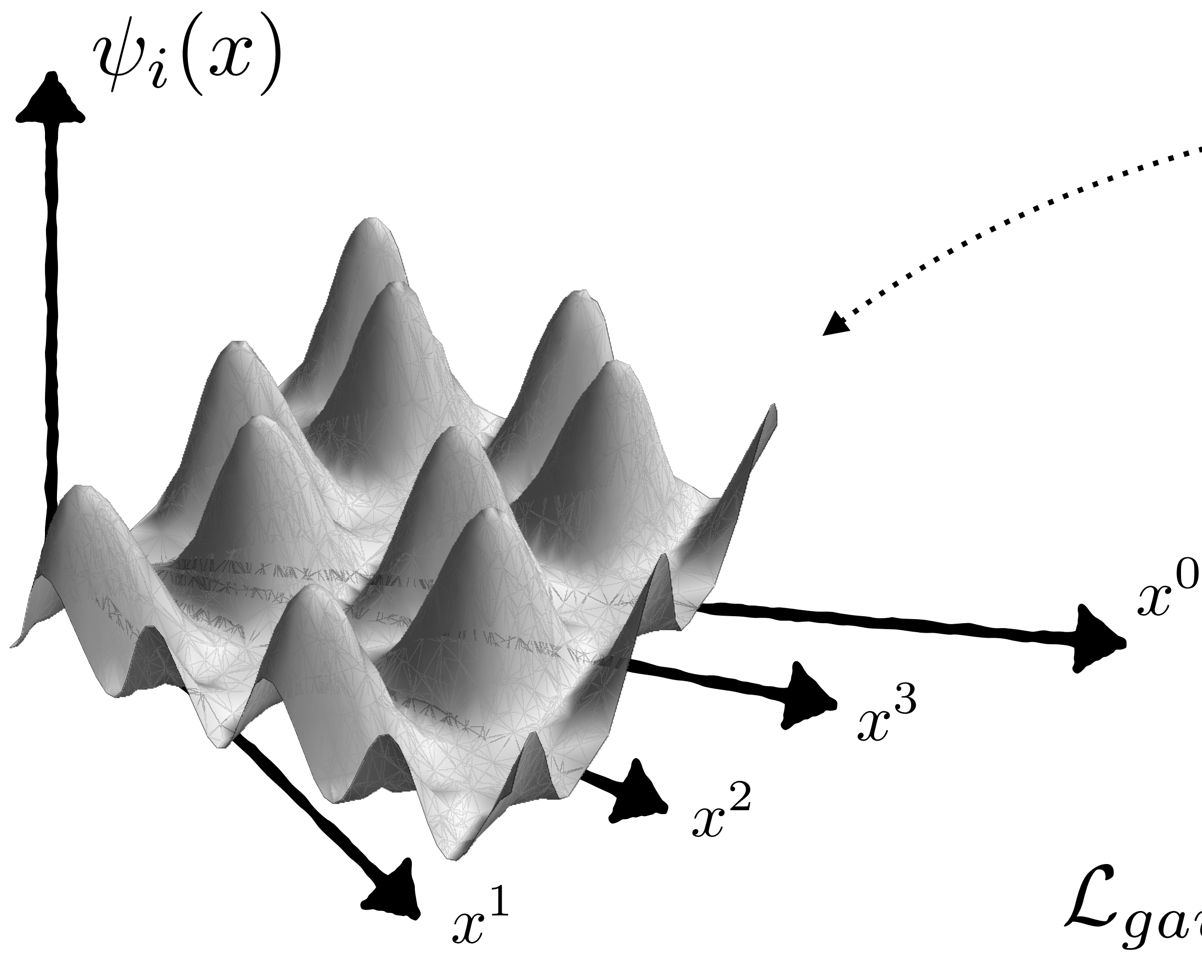
All properties come from the anticommutation relation:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \times \mathbb{1}_{n\times n}$$

Hermiticity relation:

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$$

Gauge invariance



non-local phase rotation

$$\psi(x) \rightarrow e^{i\alpha(x)}\psi(x)$$

Take the Dirac Lagrangian

$$\mathcal{L}_{Dirac} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

It is not invariant under this rotation!!!

$$\mathcal{L}_{gauge\ inv.} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu$$

QED Lagrangian

New term compensates phase rotation

$$\mathcal{L}_{QED} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}F^2_{\mu\nu} - e\bar{\psi}\gamma^\mu\psi A_\mu$$

$$A_\mu \rightarrow A_\mu - \frac{1}{e}\partial_\mu\alpha(x)$$

General principle (gauge invariance) determines the structure of the QED Lagrangian

What about QCD?

QCD gauge transformation

$$\psi(x) \to e^{i\alpha(x)}\psi(x) \longrightarrow \psi(x) \to e^{i\alpha^a(x)t^a}\psi(x)$$

QED gauge transformation

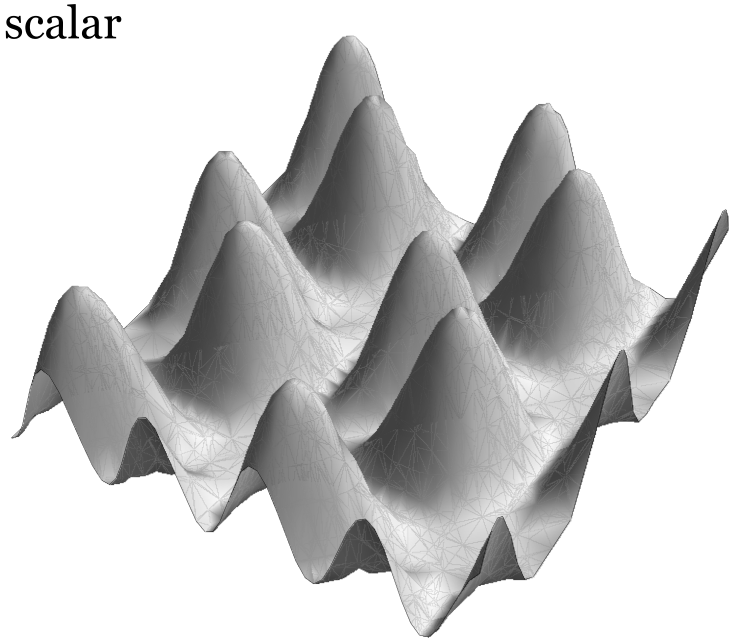
SU(3) gauge group

The problem is to construct a Lagrangian, which is invariant under this transformation

Vector in the spinor space

$$\psi(x) \longrightarrow \psi_i = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \end{pmatrix}$$

Color scalar

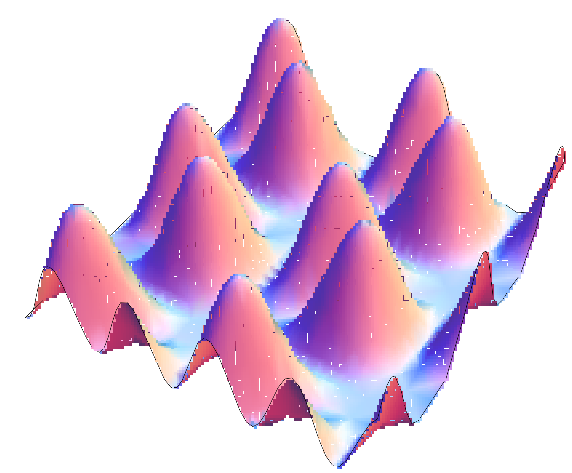


Vector in the color space

Color index (not confuse it with Dirac index)

Each element is a vector in the spinor space

Can construct color scalars (baryons and mesons)



QCD Lagrangian: Lee group

$$\psi(x) \to e^{i\alpha^a(x)t^a}\psi(x) \equiv U\psi(x)$$

generator

SU(3) gauge transformation

Properties (we should be able to construct hadrons):

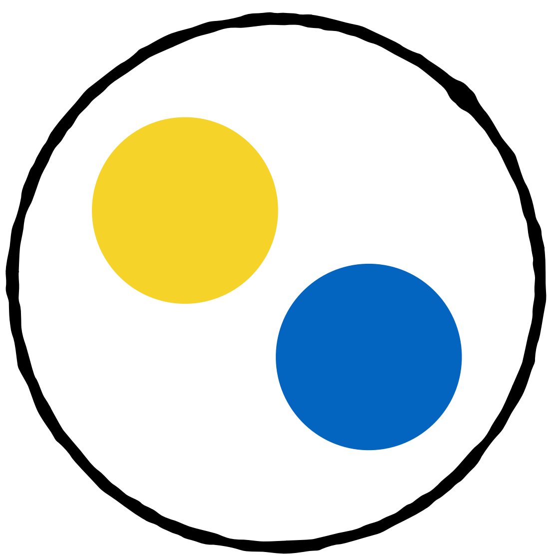
$N^2 - 1$ conditions

$$U^\dagger U = 1$$

$$\det U = 1$$

Lee group

(Unitarity)

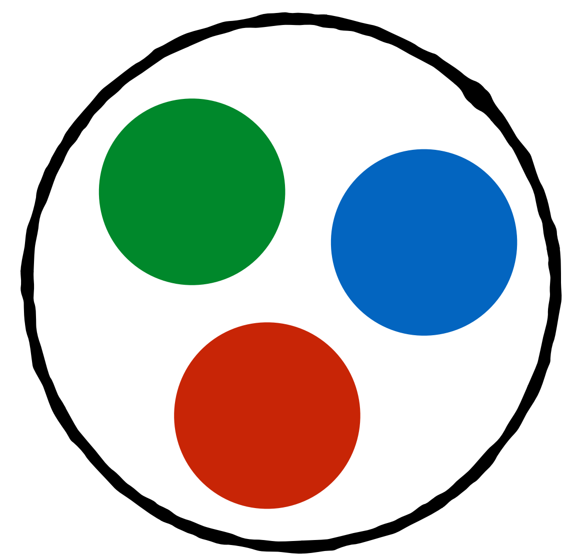


defined by $N^2 - 1$ generators

The generators "cover" all transformations

$$[t^a, t^b] = i f^{abc} t^c$$

Non-Abelian gauge theory!!!



$$\sum_i \psi_i^* \psi_i$$

$$\sum_{ijk} \epsilon_{ijk}\psi_i\psi_j\psi_k$$

All non-trivial properties come from here

QCD Lagrangian

$$\mathcal{L}_{QED} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}F_{\mu\nu}^2 - e\bar{\psi}\gamma^\mu\psi A_\mu$$

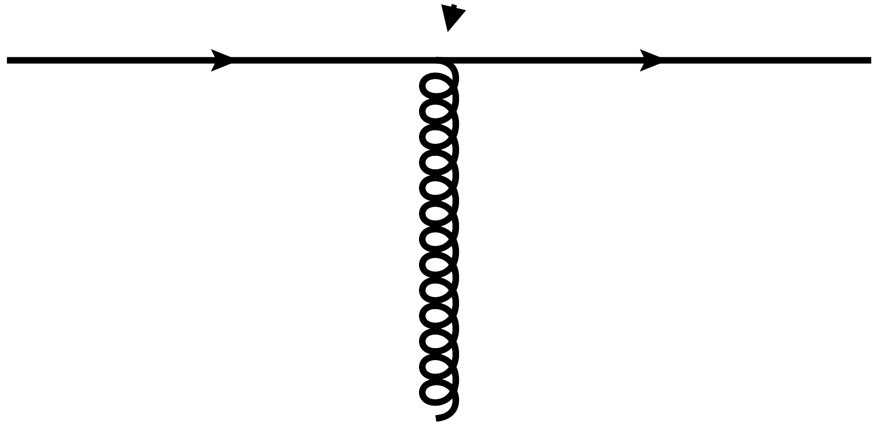
Free propagation doesn't change the color

Apply SU(3) rotation and restore gauge invariance

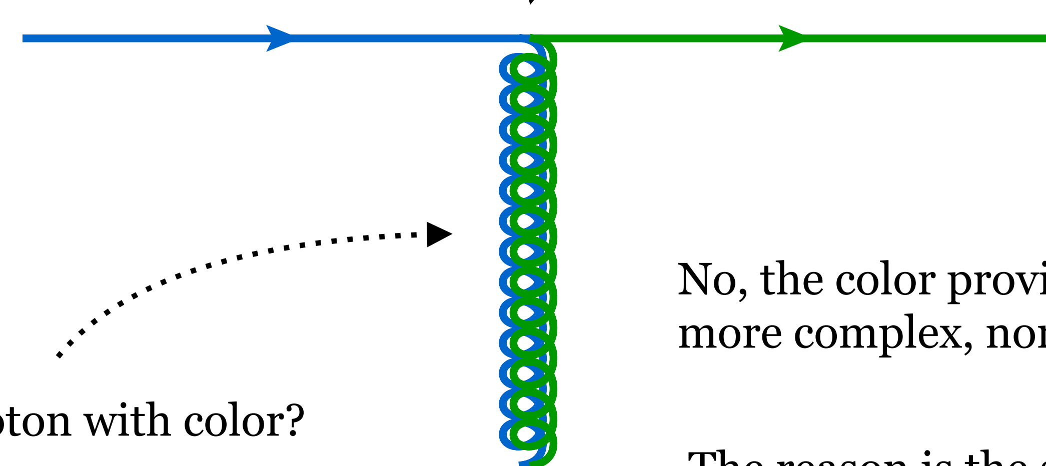
$$\mathcal{L}_{gauge\ inv.} = \bar{\psi}_i(i\partial\!\!\!/ - m)\psi_i + g\bar{\psi}_i\gamma^\mu t^a_{ij}\psi_j A^a_\mu$$

Interaction through color

$-ie\gamma^\mu$



$ig\gamma^\mu t^a$



No, the color provides significantly more complex, non-trivial dynamics

Gluon is a photon with color?

The reason is the structure of the SU(3) generators

QCD Lagrangian: free gluon part

Fundamental representation (3x3 color matrix)

$$\mathcal{L}_{gauge\ inv.} = \bar{\psi}_i(i\partial\!\!\!/ - m)\psi_i + g\bar{\psi}_i\gamma^\mu t^a_{ij}\psi_j A^a_\mu$$

$$[t^a, t^b] = if^{abc}t^c$$

structure constant

$$A^a_\mu \to A^a_\mu + \frac{1}{g}\partial_\mu\alpha^a + f^{abc}A^b_\mu\alpha^c$$

QED part

Non-abelian part

We need to find a free part which is invariant under this transformation

$$\mathcal{L}^{photon}_{QED} = -\frac{1}{4}F^2_{\mu\nu}$$

$$F^{QED}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\mathcal{L}^{gluon}_{QCD} = -\frac{1}{4}F^{a2}_{\mu\nu}$$

Color index

$$F^{aQCD}_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$$

Self-interaction term!!!

QCD Lagrangian

$$\mathcal{L}_{QCD}^{gluon} = -\frac{1}{4}F_{\mu\nu}^{a2}$$

$$F_{\mu\nu}^{aQCD} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c$$

Invariant

substitute $A_\mu^a \rightarrow A_\mu^a + \frac{1}{g}\partial_\mu \alpha^a + f^{abc}A_\mu^b \alpha^c$

Compare

$$\psi_i(x) \rightarrow e^{i\alpha^c(x)t_{ij}^c}\psi_j(x) \qquad F_{\mu\nu}^a \rightarrow e^{i\alpha^c(x)T_{ab}^c}F_{\mu\nu}^b \qquad \left(\text{where } T_{ab}^c = if^{acb}\right)$$

fundamental representation of the SU(3) group

adjoint representation of the SU(3) group

$$\mathcal{L}_{QCD} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}F_{\mu\nu}^{a2} + g\bar{\psi}\gamma^\mu t^a \psi A_\mu^a$$

Invariant under SU(3) gauge transformation

Interaction in QCD

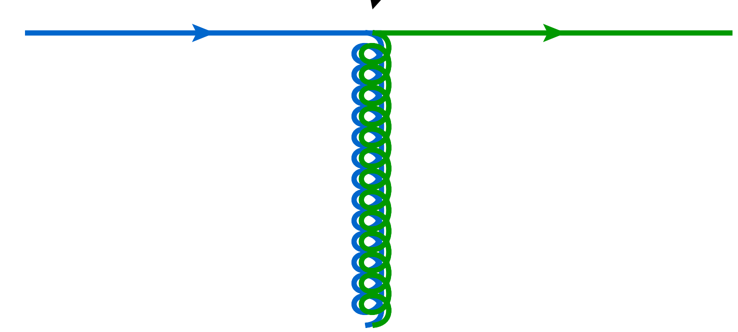
$$\mathcal{L}_{QCD} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}F_{\mu\nu}^{a2} + g\bar{\psi}\gamma^\mu t^a \psi A_\mu^a$$

Free quark propagation

Substitute $F_{\mu\nu}^a$

Quark-gluon interaction

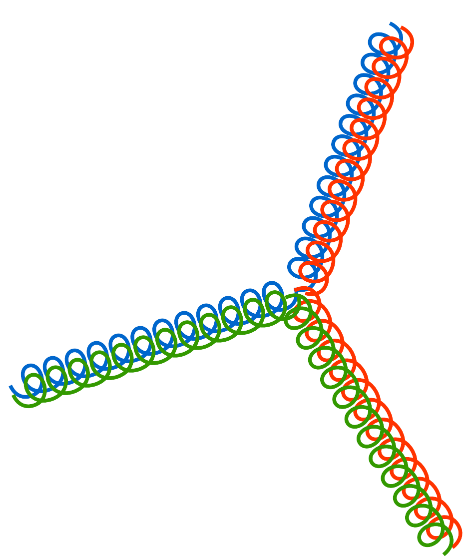
$ig\gamma^\mu t^a$

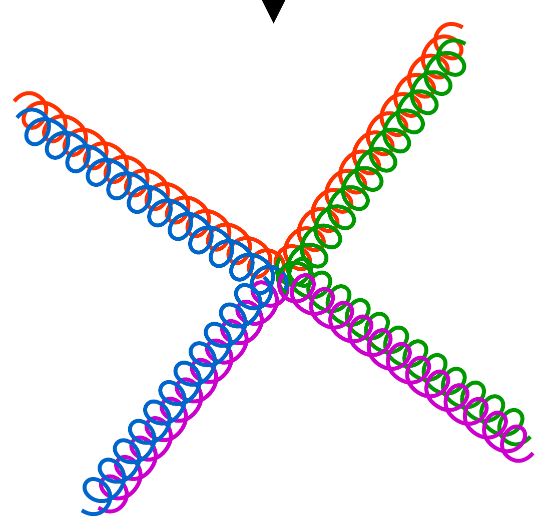


$$-\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 - gf^{abc}(\partial_\mu A_\nu^a)A^{\mu b}A^{\nu c} - \frac{1}{4}g^2(f^{eab}A_\mu^a A_\nu^b)(f^{ecd}A^{\mu c}A^{\nu d})$$

QED part (free gluon propagation)

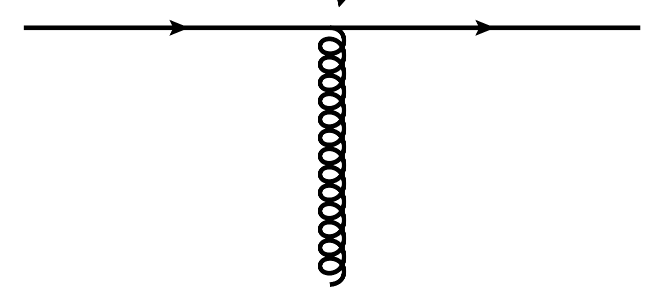
We don't have this interaction in QED!!!





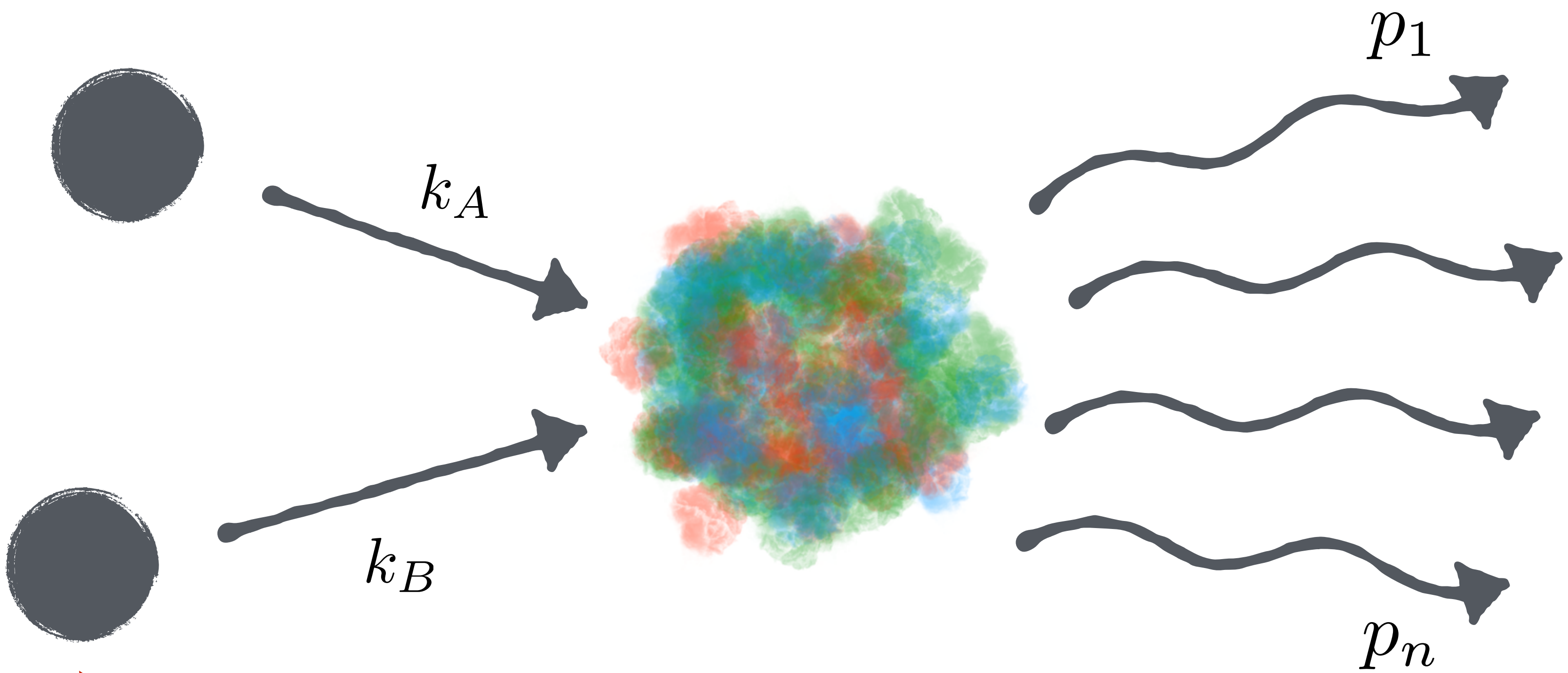
Compare with QED

$-ie\gamma^\mu$



Scattering reaction

calculate *probability* of the final state
(we discuss *quantum* theory)



fix initial state

probability of transition

$$\mathcal{P}(AB \to 1 \dots n) = \left(\prod_i \frac{d^3 p_i}{(2\pi)^3} \frac{1}{2E_i} \right) |_{out}\langle p_1 \dots p_n | AB \rangle_{in}|^2$$

S-matrix element

$$S = 1 + iT$$

transition matrix
(isolate interaction)

$$iT = (2\pi)^4 \delta^{(4)}(k_A + k_B - \sum p_i) iM$$

transition amplitude (isolate total
momentum conservation)

Using this probability one
can define the cross section

flux factor

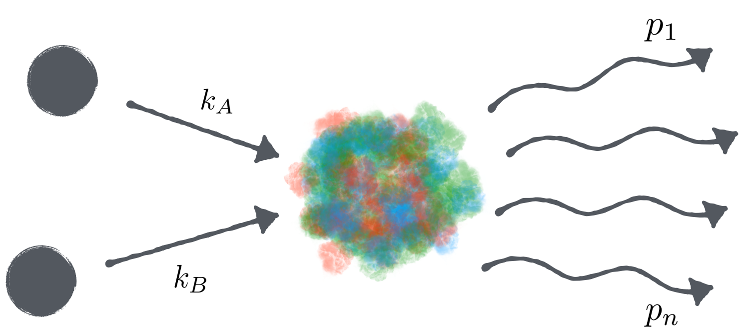
$$d\sigma = \frac{1}{4}[(k_A \cdot k_B)^2 - m_A^2 m_B^2]^{-1/2}$$
$$\times \prod_i \int \frac{d^3 p_i}{(2\pi)^3 2E_i} |M|^2 (2\pi)^4 \delta^4(k_A + k_B - \sum p_i)$$

calculate probability

Calculation of the transition amplitude

$$\mathcal{L}_{QCD} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - g f^{abc}(\partial_\mu A^a_\nu)A^{\mu b}A^{\nu c} - \frac{1}{4}g^2(f^{eau}A^a_\mu A^b_\nu)(f^{end}A^{\mu c}A^{\nu d}) + g\bar{\psi}\gamma^\mu t^a \psi A^a_\mu$$

Interaction part $\mathcal{L}_I$



How can we connect these two parts?

Expand this exponent and get the perturbation theory

$$\langle p_1 \dots p_n | iT | k_A k_B \rangle = \langle p_1 \dots p_n | T \left(\exp\left[i \int d^4x \mathcal{L}_I(x) \right] \right) | k_A k_B \rangle_0$$

T-product (we talk about operators)

two quark-gluon vertexes

two three-gluon vertexes

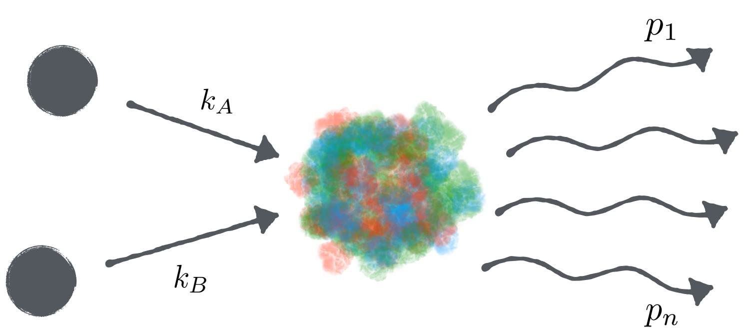
$$\langle p_1 p_2 | iT | k_A k_B \rangle$$
$$= \frac{g^4}{4!} \langle p_1 p_2 | T \Big\{ i \int d^4x \bar{\psi}\gamma^\mu t^a \psi A^a_\mu(x) \times i \int d^4z_1 f^{abc}(\partial_\mu A^a_\nu)A^{\mu b}A^{\nu c}(z_1)$$
$$\times i \int d^4z_2 f^{abc}(\partial_\mu A^a_\nu)A^{\mu b}A^{\nu c}(z_2) \times i \int d^4y \bar{\psi}\gamma^\mu t^a \psi A^a_\mu(y) \Big\} | k_A k_B \rangle$$

Note that color here is not the "color"!

Calculation of the transition amplitude

$$\mathcal{L}_{QCD} = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - g f^{abc}(\partial_\mu A^a_\nu)A^{\mu b}A^{\nu c} - \frac{1}{4}g^2(f^{eau}A^a_\mu A^b_\nu)(f^{end}A^{\mu c}A^{\nu d}) + g\bar{\psi}\gamma^\mu t^a \psi A^a_\mu$$

Interaction part $\mathcal{L}_I$



How can we connect these two parts?

Expand this exponent and get the perturbation theory

$$\langle p_1 \dots p_n | iT | k_A k_B \rangle = \langle p_1 \dots p_n | T \left(\exp \left[i \int d^4x \mathcal{L}_I(x) \right] \right) | k_A k_B \rangle_0$$

T-product (we talk about operators)

An infinite number of combinations. Higher order of expansion are suppressed by powers of the coupling constant (we hope)

four quark-gluon vertexes

$$\langle p_1 p_2 | iT | k_A k_B \rangle$$
$$= \frac{g^4}{4!} \langle p_1 p_2 | T \Big\{ i \int d^4x \bar{\psi}\gamma^\mu t^a \psi A^a_\mu(x) \times i \int d^4z_1 \bar{\psi}\gamma^\nu t^b \psi A^b_\nu(z_1)$$
$$\times i \int d^4z_2 \bar{\psi}\gamma^\rho t^c \psi A^c_\rho(z_2) \times i \int d^4y \bar{\psi}\gamma^\sigma t^d \psi A^d_\sigma(y) \Big\} | k_A k_B \rangle$$

Note that color here is not the "color"!

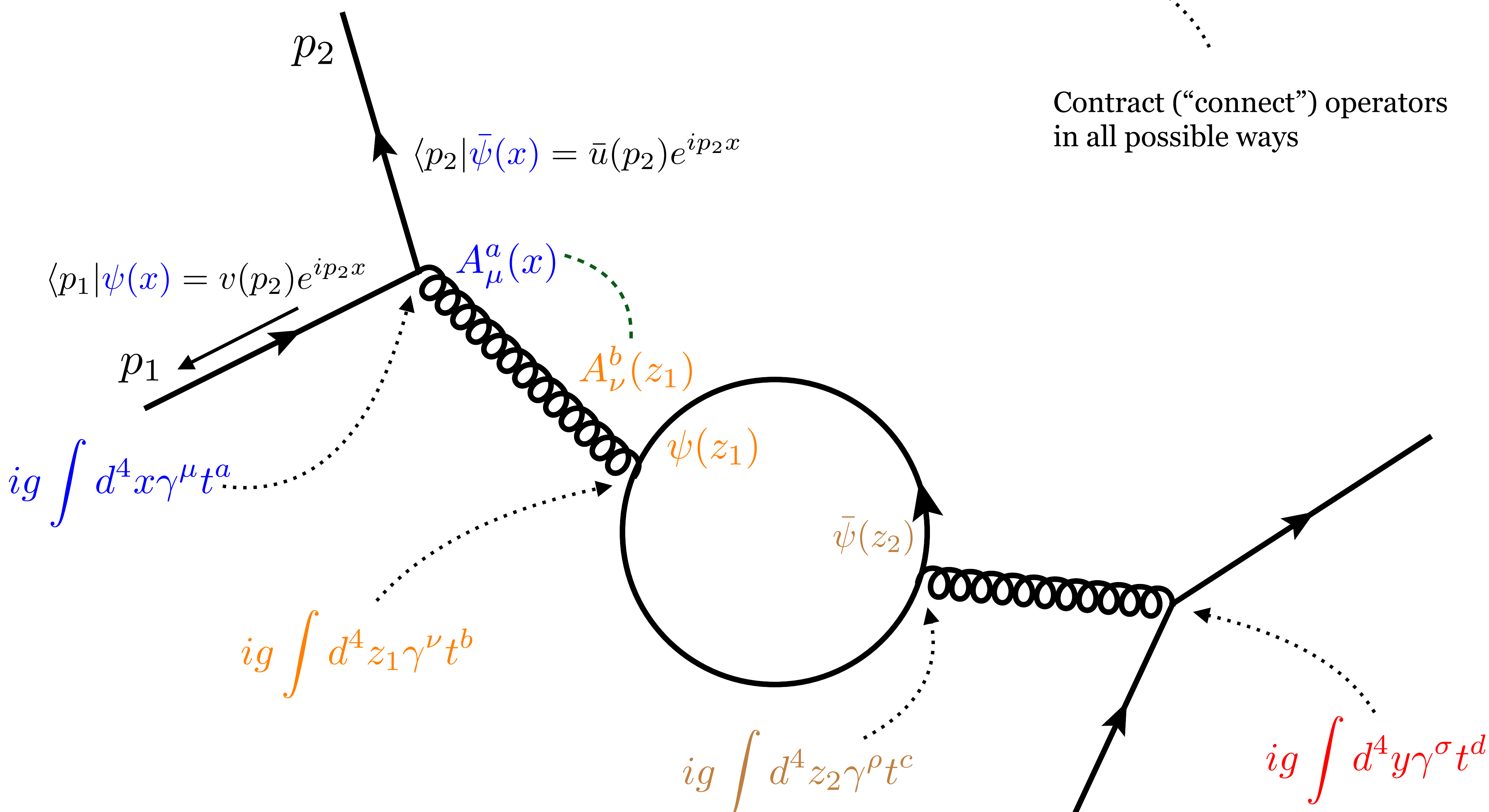
Contraction of operators

One explicit contraction

$$\langle p_1 p_2 | iT | k_A k_B \rangle$$
$$= \frac{g^4}{4!} \langle p_1 p_2 | T \Big\{ i \int d^4x \bar{\psi} \gamma^\mu t^a \psi A^a_\mu(x) \times i \int d^4z_1 \bar{\psi} \gamma^\nu t^b \psi A^b_\nu(z_1)$$
$$\times i \int d^4z_2 \bar{\psi} \gamma^\rho t^c \psi A^c_\rho(z_2) \times i \int d^4y \bar{\psi} \gamma^\sigma t^d \psi A^d_\sigma(y) \Big\} | k_A k_B \rangle$$

That is how we get Feynman diagrams

Contract ("connect") operators in all possible ways

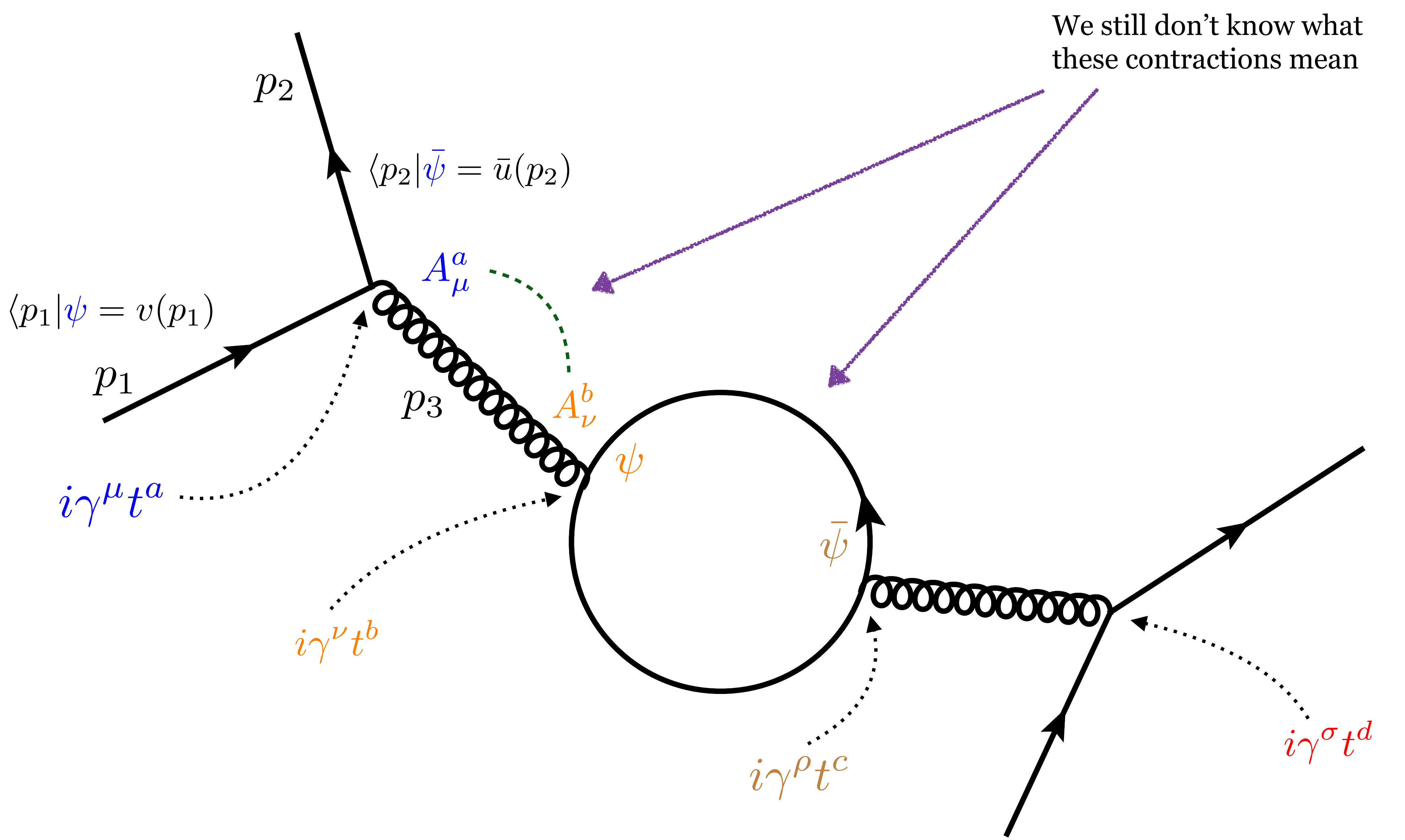


Contraction of operators (momentum representation)

We can integrate over coordinates and extract momentum conservation

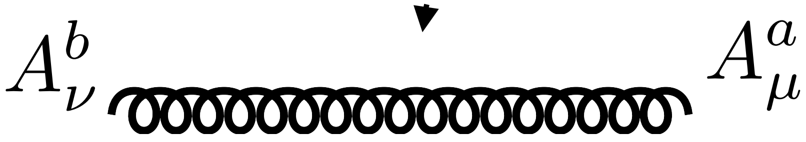
$$\int d^4x e^{ip_1x}e^{ip_2x}e^{-ip_3x} = (2\pi)^4\delta^4(p_1+p_2-p_3)$$

$$iT \to iM$$



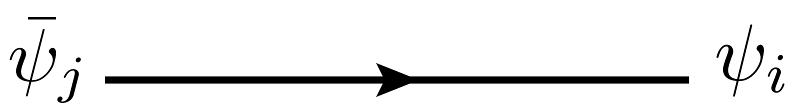
QCD propagators

Gluon propagator



By definition!!!

$$= \langle 0 | T\{ A^a_\mu(x) A^b_\nu(y) \} | 0 \rangle$$



$$= \langle 0 | T\{ \psi_i(x) \bar{\psi}_j(y) \} | 0 \rangle$$

assume color index as well

Quark propagator

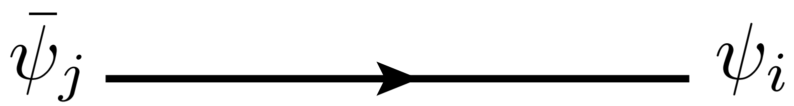
$$T\{\psi_i(x)\bar{\psi}_j(y)\} \equiv \theta(x^0 - y^0)\psi_i(x)\bar{\psi}_j(y) - \theta(y^0 - x^0)\bar{\psi}_j(y)\psi_i(x)$$

apply to this definition

$$\begin{cases} (i\gamma^\mu \partial_\mu - m)\psi(x) = 0 \\ \{\psi_i(x), \psi_j^\dagger(y)\} = \delta^{(3)}(x - y)\delta_{ij} \end{cases}$$

Can obtain equation for this object from the free part of the QCD Lagrangian

Quark propagator



Solution of this equation

Let's solve this equation

$$(i\not\partial - m)T\{\psi(x)\bar\psi(y)\} = i\delta^{(4)}(x-y)\times \mathbb{1}_{n\times n}$$

Let's find the solution in a form:

$$T\{\psi(x)\bar\psi(y)\} = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} S(p)$$

$$\int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} (\not p - m) S(p) = i\delta^{(4)}(x-y)\times \mathbb{1}_{n\times n}$$

It was easy to obtain this solution

Quark propagator in momentum representation:

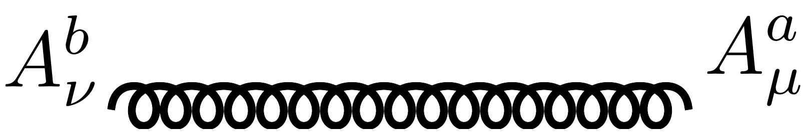
$$S(p) = \frac{i\delta_{ij}}{\not p - m + i\epsilon}$$

Color

Quark propagator in coordinate representation:

$$T\{\psi(x)\bar\psi(y)\} = \int \frac{d^4p}{(2\pi)^4} \frac{i\delta_{ij}}{\not p - m + i\epsilon} e^{-ip(x-y)}$$

Gluon propagator



We start from the free gluon Lagrangian:

$$-\frac{1}{4}F_{\mu\nu}^{a2} = \frac{1}{2}A_\mu^a(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)A_\nu^a$$

Apply

Gluon propagator is defined as:

$$D^{\mu\nu}(x-y) \equiv \langle 0|T\{A_\mu(x)A_\nu(y)\}|0\rangle$$

The equation for the gluon propagator in the coordinate representation:

$$(\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu)D^{\nu\rho}(x-y) = i\delta_\mu^\rho\delta^{(4)}(x-y)$$

The equation for the gluon propagator in the momentum representation:

Can we solve this equation?

$$(-k^2 g_{\mu\nu} + k_\mu k_\nu)D^{\nu\rho}(k) = i\delta_\mu^\rho$$

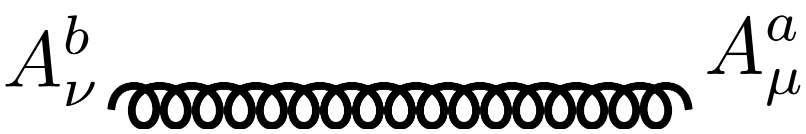
We can not construct solution of the equation using this method!

Singular 4x4 matrix!

$$\det A = 0$$

Functional integral

$$D^{\mu\nu}(x-y) \equiv \langle 0|T\{A_\mu(x)A_\nu(y)\}|0\rangle$$



Can we define this differently?

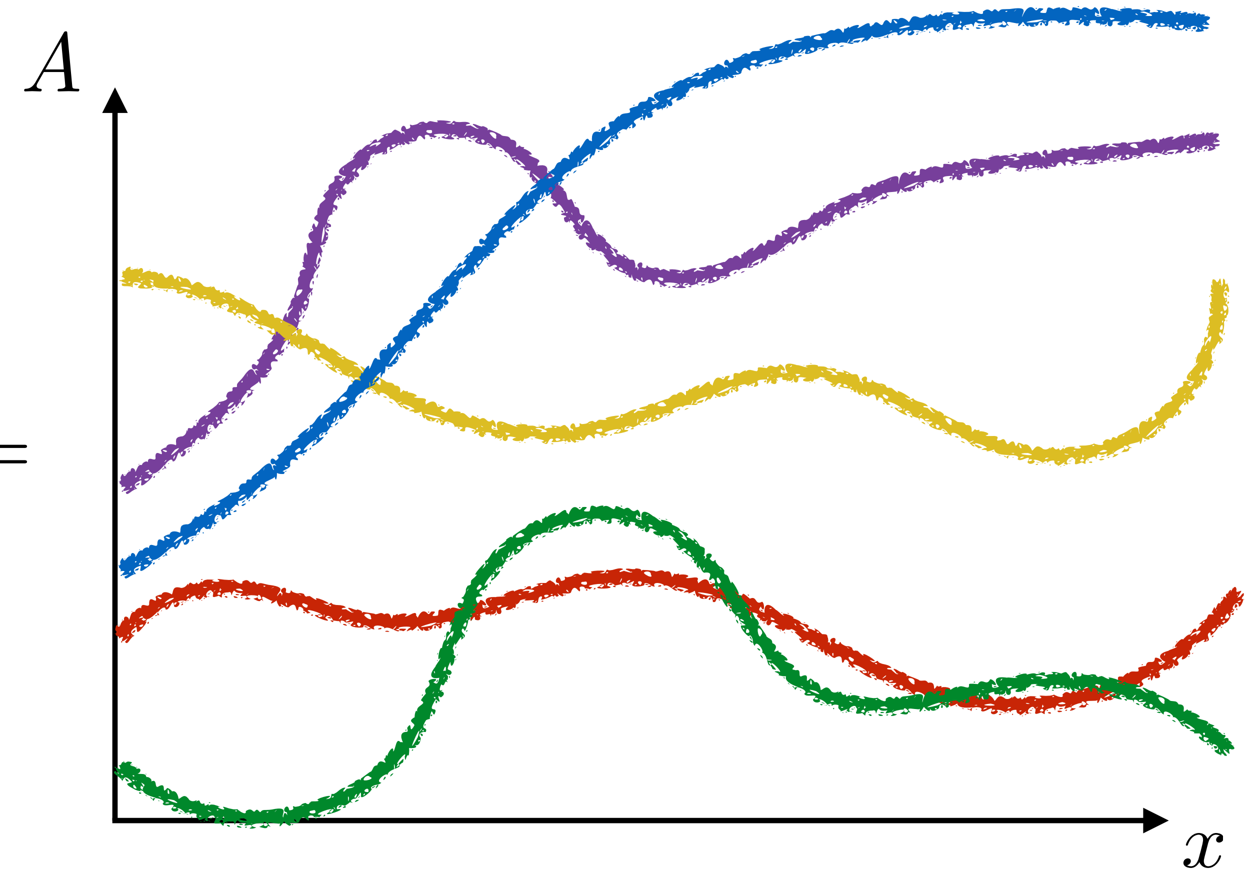
Some operator

All possible classical fields

Lagrangian with interaction

$$\langle\Omega|T\mathcal{O}(\phi)|\Omega\rangle = \frac{\int \mathcal{D}\phi\mathcal{O}(\phi)\exp\left[i\int d^4x\mathcal{L}\right]}{\int \mathcal{D}\phi\exp\left[i\int d^4x\mathcal{L}\right]}$$

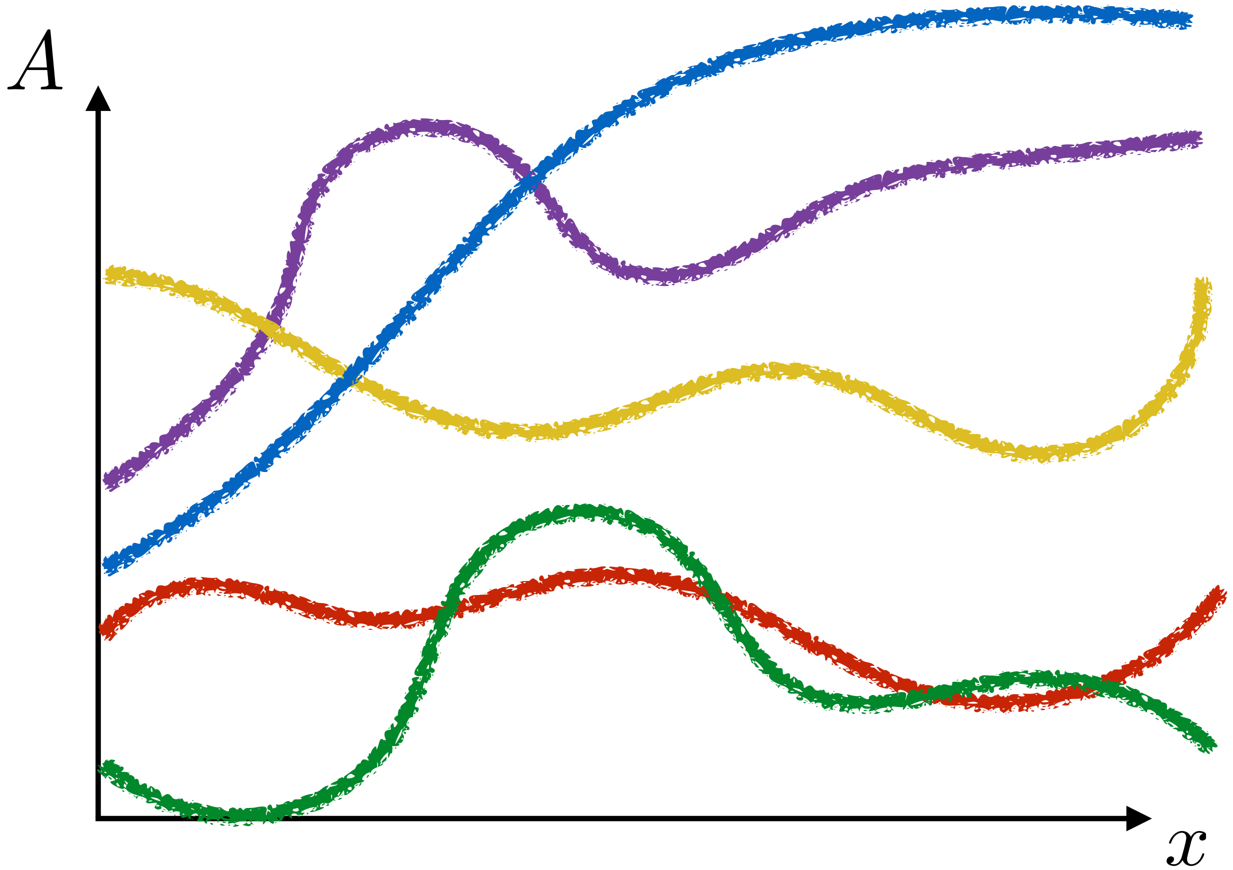
$$\langle\Omega|T\mathcal{O}(A)|\Omega\rangle =$$



Functional representation of the gluon propagator

$$\langle 0|T\{A_\mu(x)A_\nu(y)\}|0\rangle = \frac{\int \mathcal{D}A \; A_\mu(x)A_\nu(y)\exp\left[i\int d^4x\mathcal{L}_{\rm free}\right]}{\int \mathcal{D}A \exp\left[i\int d^4x\mathcal{L}_{\rm free}\right]}$$

Integrate over infinite number of equal combinations



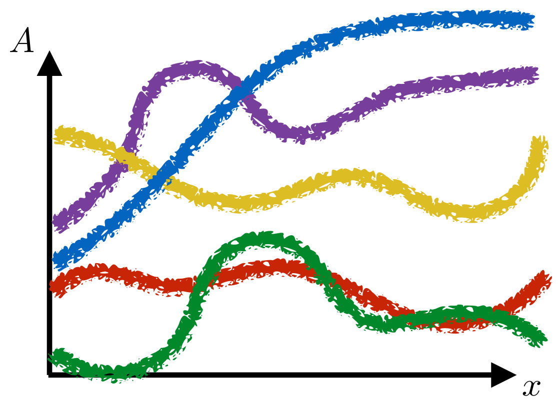
$$A_\mu^a \to A_\mu^a + \frac{1}{g}\partial_\mu\alpha^a + f^{abc}A_\mu^b\alpha^c$$

Gauge transformation

This integration leads to infinity!!!

Can we separate it?

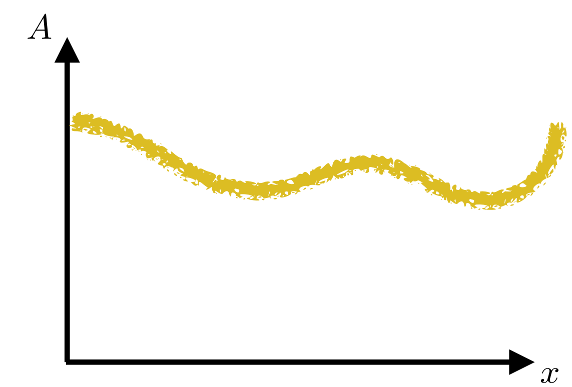
Functional representation of the gluon propagator



$$\langle\Omega|T\mathcal{O}(A)|\Omega\rangle = \frac{\int \mathcal{D}A\ \mathcal{O}(A)\exp\left[i\int d^4x\mathcal{L}\right]}{\int \mathcal{D}A\ \ \exp\left[i\int d^4x\mathcal{L}\right]}$$

Integrate over fixed configuration

$$\langle\Omega|T\mathcal{O}(A)|\Omega\rangle = \frac{\int \mathcal{D}A\ \mathcal{O}(A)\exp\left[i\int d^4x[\mathcal{L} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2]\right]\times G}{\int \mathcal{D}A\ \ \exp\left[i\int d^4x[\mathcal{L} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2]\right]\times G}$$

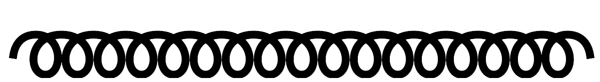


Gauge fixing term

Integral is now well defined

$$\left(-k^2 g_{\mu\nu} + (1-\frac{1}{\xi})k_\mu k_\nu\right) D^{\nu\rho}(k) = i\delta^\rho_\mu$$

$$D^{\mu\nu}(k) = \frac{-i}{k^2+i\epsilon}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right)$$



Functional representation of the gluon propagator

$$D^{ab}_{\mu\nu}(k) = \frac{-i\delta^{ab}}{k^2 + i\epsilon}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)$$

Landau gauge: $\xi = 0$

$$D^{ab}_{\mu\nu}(k) = \frac{-i\delta^{ab}}{k^2 + i\epsilon}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)$$

Feynman gauge: $\xi = 1$

We are going to use this

$$D^{ab}_{\mu\nu}(k) = \frac{-ig_{\mu\nu}\delta^{ab}}{k^2 + i\epsilon}$$

Axial gauge

$$-\frac{1}{\xi}(n^\mu A_\mu)^2$$

Arbitrary vector

Other forms of the gauge fixing term are possible

Light-cone gauge

$$D^{ab}_{\mu\nu}(k) = \frac{-i\delta^{ab}}{k^2 + i\epsilon}\left(g_{\mu\nu} - \frac{n_\mu k_\nu + n_\nu k_\mu}{n \cdot k} - \frac{(n^2 + \xi k^2)k_\mu k_\nu}{(n \cdot k)^2}\right)$$

$$n^2 = 0$$
$$\xi = 0$$

Faddeev-Popov ghosts

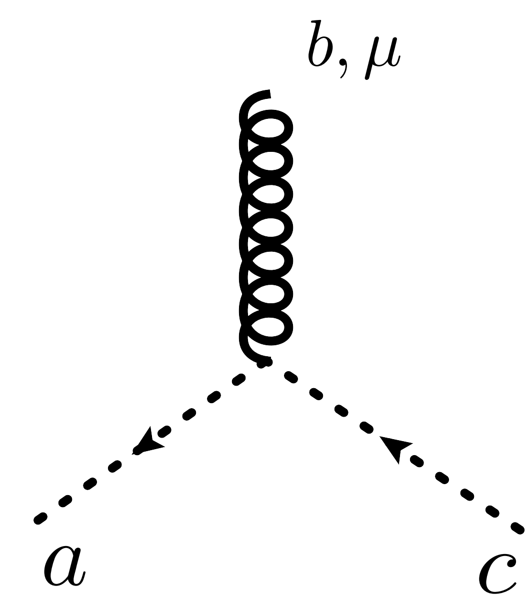
$$\langle \Omega | T\mathcal{O}(A) | \Omega \rangle = \frac{\int \mathcal{D}A \; \mathcal{O}(A) \exp\left[i \int d^4x [\mathcal{L} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2] \right] \times G}{\int \mathcal{D}A \; \exp\left[i \int d^4x [\mathcal{L} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2] \right] \times G}$$

It is not for free!

We have to introduce a new class of pseudo-real field - ghosts

$$G \equiv \int \mathcal{D}c \mathcal{D}\bar{c} \exp\left[i \int d^4x \mathcal{L}_{\mathrm{ghost}} \right]$$

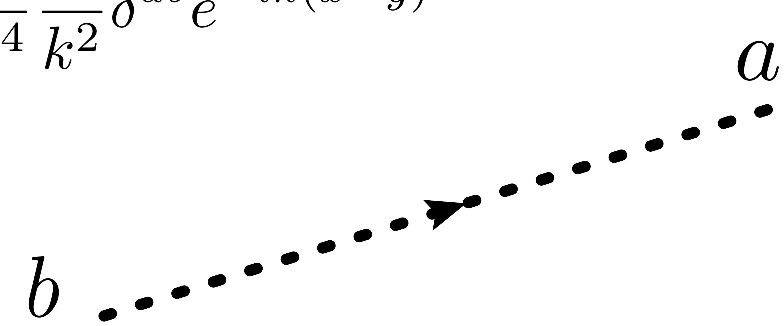
$$\mathcal{L}_{\mathrm{ghost}} = \bar{c}^a (-\partial^2 \delta^{ac} - g \partial^\mu f^{abc} A^b_\mu) c^c$$



$-g f^{abc} p^\mu$

Ghost propagator

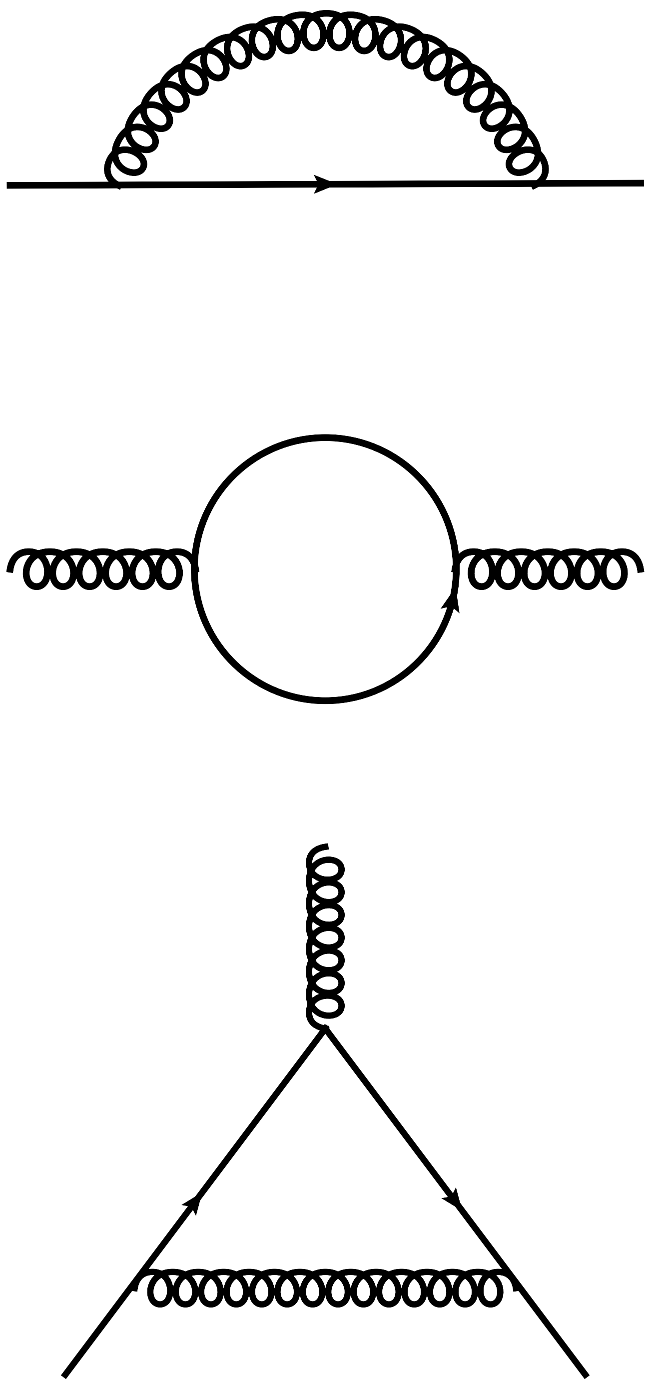
$$\langle c^a(x) \bar{c}^b(y) \rangle = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2} \delta^{ab} e^{-ik(x-y)}$$



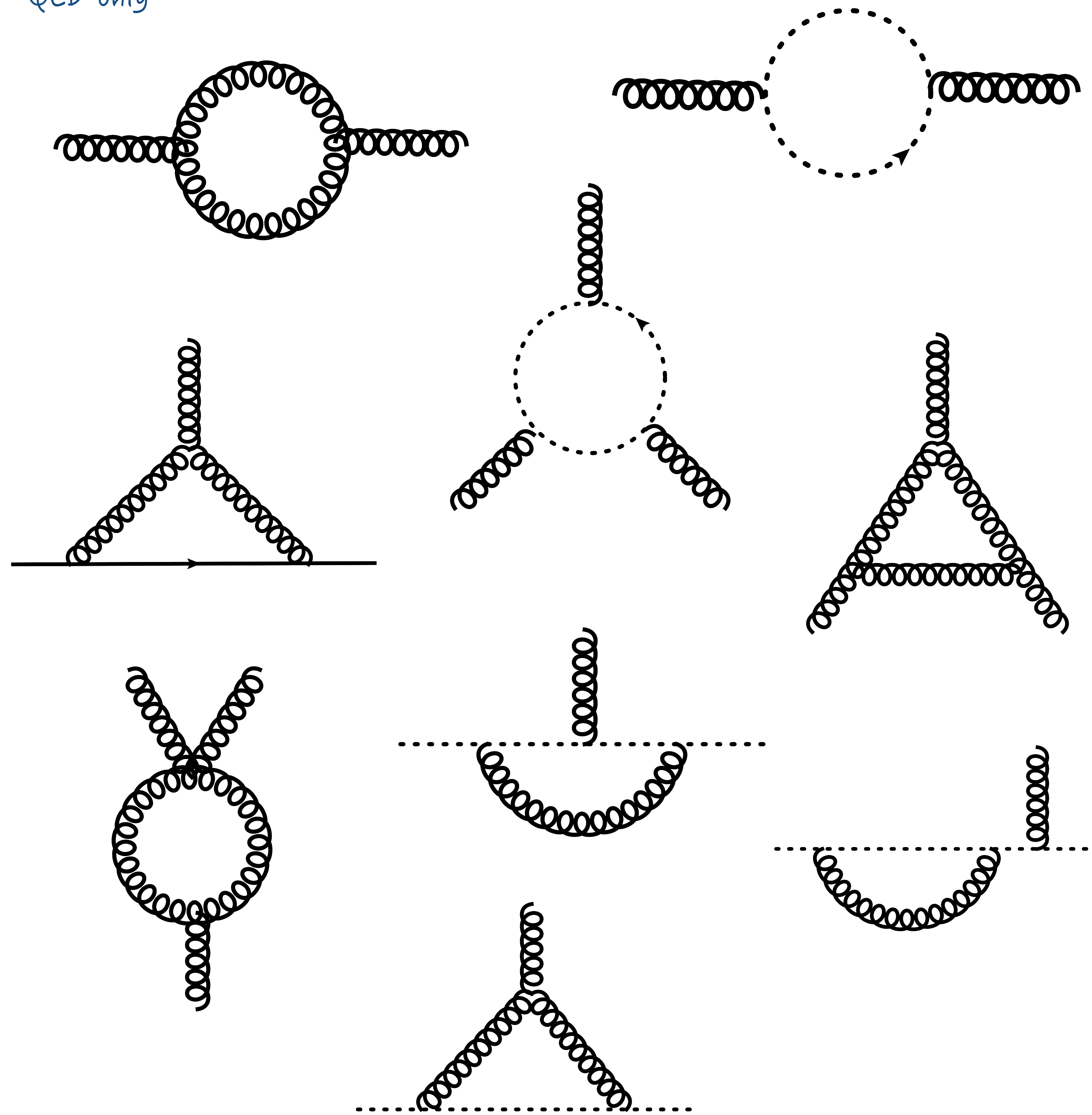
Divergent diagrams in QED and QCD

Infinity!!!

QED and QCD



QCD only



Renormalization

$$\phi_0 = Z_\phi^{1/2} \phi_R$$

"Bare field" is infinite

"Renormalized field" is finite

Renormalization constant is infinite

We assume that infinity in diagrams can be absorbed by infinity in fields, masses and coupling constants

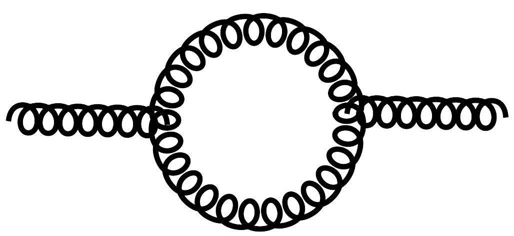
The same relations for mass and renormalization constant

$$m_0 = Z_m^{1/2} m_R$$

$$g_0 = Z_g g_R \mu^\epsilon$$

Arbitrary mass parameter

We should be able to separate divergence from the diagram. Usually we use dimensional regularization



1

$$\int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2(p-k)^2} \to \int \frac{d^{4-2\epsilon}p}{(2\pi)^4} \frac{1}{p^2(p-k)^2}$$

2

$$\frac{1}{p^2(p-k)^2} = \int_0^1 dx \frac{1}{\left[xp^2 + (1-x)(p-k)^2\right]^2}$$

Feynman parameters

3 Change of variables and Wick rotation

4 Integration in d dimensions

5 Extract divergence

Beta function

Consider a coupling scale at a given scale

$$g_0 = Z_g g_R \mu^\epsilon$$

Renormalized coupling is a function of the mass scale parameter

$$\beta(g_R, m_R) = \mu \frac{\partial}{\partial \mu} g_R(\mu)$$

The observable shouldn't depend on this parameter, see renormalization group equation

QED beta function

$$\beta(e) = \frac{e^3}{12\pi^2}$$

QCD beta function

$$\beta(g) = -\frac{g^3}{(4\pi)^2}\left[\frac{11}{3}N_c - \frac{2}{3}n_f\right]$$

The sign of this two functions is different

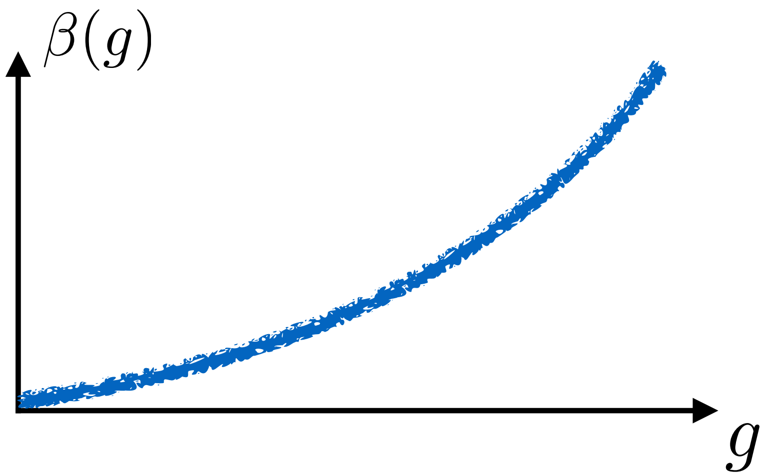
We can define the coupling constant at another scale:

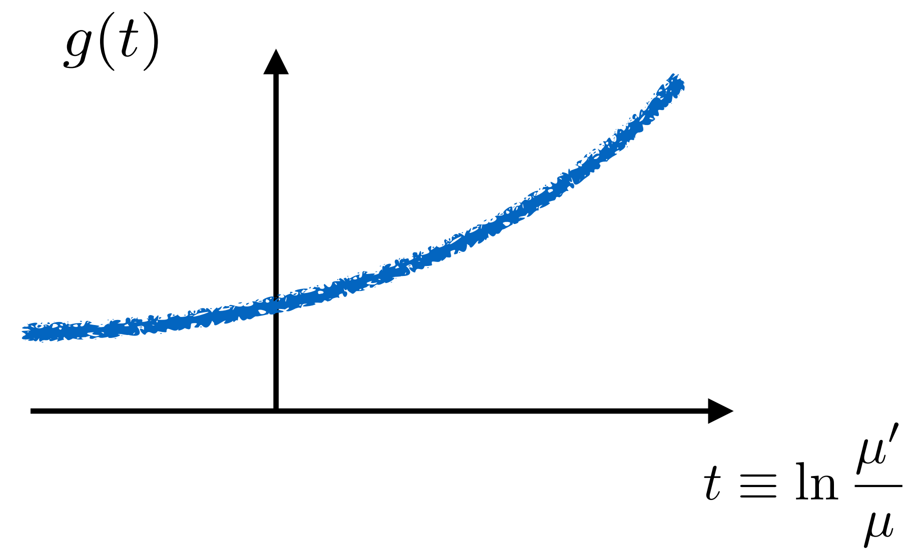
$$\frac{d}{d\log(\mu'/\mu)} g' = \beta(g')$$

This equation should define dependence of the coupling constant on the scale

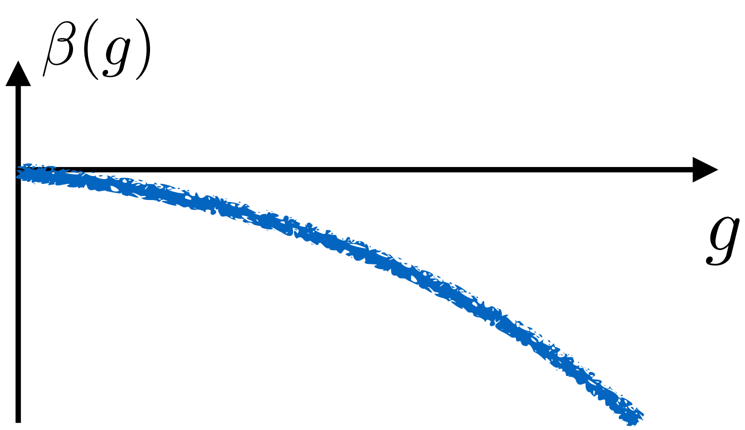
The sign of the beta function

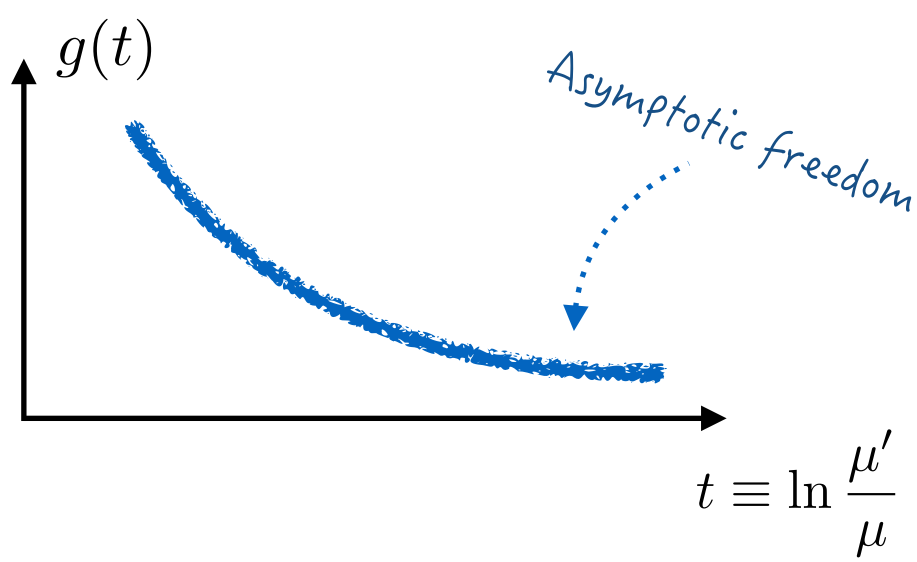
QED





QCD





$$\alpha_s(\mu') = \frac{\alpha_s(\mu)}{1 + \{b_0 \alpha_s(\mu)/2\pi\} \log(\mu'/\mu)}$$

Running of the QCD coupling constant

Running of the coupling constant